Self-assembling of networks in an agent-based model

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We propose a model to show the self-assembling of networklike structures between a set of nodes without using preexisting positional information or long-range attraction of the nodes. The model is based on Brownian agents that are capable of producing different local (chemical) information and respond to it in a nonlinear manner. They solve two tasks in parallel: (i) the detection of the appropriate nodes, and (ii) the establishment of stable links between them. We present results of computer simulations that demonstrate the emergence of robust network structures and investigate the connectivity of the network by means of both analytical estimations and computer simulations.

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I. INTRODUCTION

The *emergence* of network structures, i.e., the *self-organized* formation of links between a set of nodes is of crucial importance in many different fields. In electronic engineering, for instance, one is interested in the *self-assembling* and *self-repairing* of electronic circuits [1–3], while in biology models for the self-wiring of *neuronal networks* are investigated [4,5]. On the social level, the self-organization of human trail networks between different destinations is a similar problem [6]. Also the establishment of connections on demand in telecommunication or logistics is related to the problem discussed here.

A desirable feature of self-organized networks is their *adaptivity*. This means that new nodes can be linked to the existing network or linked nodes can be disconnected from the network if this is required, e.g., by the change of some external conditions. It is worth noting that such a behavior should not be governed by a ''supervisor'' or ''dispatcher,'' it should rather result from the adaptive capabilities of the network itself.

Such problems become even more complicated if no *a priori* information about the network structure is provided, i.e., the network has to self-organize itself not only regarding the links but also regarding the nodes. This is the case, for instance, if the nodes to be linked to the network are ''unknown'' in the sense that they *first* have to be *discovered* and only *then* can be *connected*. A common biological example is the formation of a trail system in ants to connect a nest to a set of food sources that first have to be found [7,8]. Such networks are known to be rather flexible and adaptive. After food sources are exhausted, they are ''disconnected'' from the existing network, because they are no longer visited and the respective trail is no longer maintained, but newly found food sources can be linked to the existing network as well.

Another important example of this kind of phenomena can be found in the self-wiring of neural structures. A neuron that grows from the retina of the eye towards the optic tectum (or superior colliculus) of the brain, does not ''know'' from the outset about its destination node in the brain, hence it has to navigate through an unknown environment in order to detect and to reach the appropriate area. Neural growth cones appear to be guided by at least four different mechanisms: contact attraction, chemoattraction, contact repulsion, and chemorepulsion [9]. These mechanisms seem to act simultaneously and in a coordinated manner to direct pathfinding. Once these specific pathways are established, neuronal growth cones can navigate over long distances to find their correct targets.

It is known that gradients of different chemical cues play a considerable role in this navigation process. They provide a kind of *positional information* for the navigation of the growth cones [10]. Already in 1963 Sperry [11] proposed that positional information might be encoded in the form of gradients of signaling molecules that could be detected by the axons. That is, axons could read positional information at every point on the tectum. It is worth noting that such an explanation assumes that the positional information resulting from the different gradients preexists in the environment. It may then provide a kind of long-range attraction or repulsion for the growth cones, which act together with other short-range mechanisms.

This points to the question that shall be answered in this paper: is it possible to link a set of nodes without using preexisting positional information or any kind of long-range attraction of the nodes? Can the process of generating positional information, i.e., the detection of ''unknown'' nodes and the estabishment of chemical gradients, *and* the process of network formation, i.e., the establishment of links between nodes, occur in parallel, on a comparable time scale, as a process of coevolution?

In order to show this, in Sec. II we introduce a model of Brownian agents that are capable of producing different local (chemical) information and respond to it in a nonlinear man-

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ner. In Sec. III, this model is applied to the formation of a network between a set of nodes. We present results of computer simulations that demonstrate the emergence of network structures. In Sec. IV, we investigate the network connectivity as a particular quantitative feature of the network by means of both analytical estimations and computer simulations. In Sec. V, we conclude the results and comment also on the agent-based method used in this paper.

II. MODEL OF BROWNIAN AGENTS

The self-organization of a network is, in the considered case, based on two different kinds of *activities*: (i) the generation of positional information in terms of chemical gradients, and (ii) the nonlinear response to the existing information in order to link the different nodes. These rather complex processes are not performed by usual physical particles, therefore we have introduced the concept of *Brownian agents* [12] as a simple way to consider certain activities within the framework of statistical physics.

A Brownian agent i is characterized by different state variables that could be either external variables such as its position $\boldsymbol{r}_i$ or its velocity $\boldsymbol{v}_i$, or *internal degrees of freedom*. θ_i, for example, is assumed to be a discrete valued parameter that allows to describe different responses of agent i to external signals, or different changes of its environment. Because all kind of activities need energy, the agent's energy depot e_i is another important internal degree of freedom [13]. These state variables may change in the course of time, either or both by deterministic and stochastic influences. Similar to the description of Brownian motion, we will use a generalized Langevin equation for the Brownian agent (which also justifies its denotation). For the change of the agent's position we may assume an overdamped Langevin equation,

$$\frac{d\boldsymbol{r}_i}{dt} = \alpha_i \frac{\partial h^e(\boldsymbol{r},t)}{\partial \boldsymbol{r}}\Big|_{\boldsymbol{r}_i,\theta_i} + \sqrt{2\varepsilon_i}\xi_i(t). \tag{1}$$

The second term denotes the stochastic influences, where $\xi_i(t)$ is white noise with $\langle \xi_i(t)\rangle = 0$ and $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$. The strength of the stochastic force ε_i could be in general an individual parameter to weight the stochastic influences, this way it can for example measure the individual *sensitivity* $\omega_i \propto 1/\varepsilon_i$ of the agent. The first term denotes the deterministic influences that are in the considered case assumed to result from the *gradient* of an *effective field* $h^e(\boldsymbol{r},t)$. This field contains the positional information provided by different chemical cues as specified below. The parameter α_i describes the strength of the individual response of the agent to the field and weights the deterministic influences. α_i can be used to describe different responses to the field.

(i) Attraction to the field, $\alpha_i > 0$, or repulsion, $\alpha_i < 0$.

(ii) Response only if the local value of the field is above a certain threshold h_0: $\alpha_i = \Theta[h^e(\boldsymbol{r},t) - h_0]$, with $\Theta[y]$ being the Heavyside function; $\Theta = 1$, if $y > 0$, otherwise $\Theta = 0$.

(iii) Response only if the agent has a specific internal value θ: $\alpha_i = \delta_{\theta_i,\theta}$.

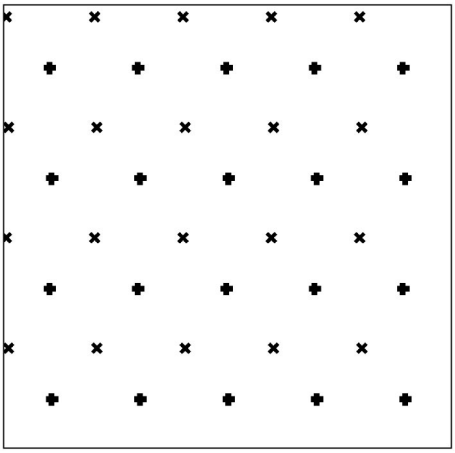

FIG. 1. Example of a regular distribution of 40 nodes on a lattice of size $A = 100 \times 100$. For the computer simulations, periodic boundary conditions have been used. $z_+ = 20$, $z_- = 20$. $\{\times\}$ indicates nodes with a potential $V_j = -1$, $\{+\}$ indicates nodes with a potential $V_j = +1$.

Throughout this paper, we assume the two individual parameters as constants: $\alpha_i \equiv \alpha = 1$ and $\varepsilon_i = D_n$ where D_n is the spatial diffusion coefficient, but we want to mention that the idea of an adjustable sensitivity has been successfully applied to model search problems with Brownian agents [8,14].

In addition to the movement of the Brownian agents, i.e., changes of their state variables $\boldsymbol{r}_i$, we also have to consider changes of their internal degree of freedom $\theta_i(t)$ that in this application should have one of the following values: $\theta_i \in \{0,-1,+1\}$. Initially, $\theta_i(t_0) = 0$ holds for every agent. The parameter θ_i can be changed in the course of time by an interaction between the moving agents and the nodes. To be specific, we consider a two-dimensional surface, where a number of $j = 1, \ldots, z$ nodes are located at the positions $\boldsymbol{r}_j^z$ (cf. Fig. 1). A number of z_+ nodes should be characterized by a positive potential, $V_j = +1$, while $z_- = z - z_+$ nodes have a negative potential, $V_j = -1$. We note explicitly that the nodes do *not* have any *long-range effect* on the agents, such as attraction or repulsion. Their effect is restricted to their location, $\boldsymbol{r}_j^z$.

It is the (twofold) task of the Brownian agents, first to *discover* the nodes and then to *link* nodes with an opposite potential, this way forming a self-organized network between the set of nodes. If an agent hits one of the nodes, its internal degree of freedom is changed according to the following equation:

$$\Delta\theta_i(t) = \int_A \sum_{j=1}^{z} (V_j - \theta_i)\frac{1}{A}\delta(\boldsymbol{r}_j^z - \boldsymbol{r}_i(t))d\boldsymbol{r}. \tag{2}$$

The δ function is equal to 1 only for $\boldsymbol{r}_j^z = \boldsymbol{r}_i$ and zero otherwise. So, Eq. (2) indicates, that an agent changes its internal state, θ_i, only if it hits one of the nodes. Then it takes over the value of the potential of the respective node, V_j, which means θ_i remains constant if $V_j = \theta_i$, and $\theta_i \to V_j$, if $V_j \neq \theta_i$. We note that the probability for a (pointlike) agent to hit a (pointlike) node is almost vanishing. However, the com-

puter simulations discussed in the following section are carried out on a discrete lattice, so the agent and the node both have a finite extension, in which case Eq. (2) makes sense.

If the Brownian agent hits one of the nodes, this impact may result in an active state of the agent—a *kick*, originated by the potential, which may change the internal parameter, θ_i, due to Eq. (2). In the active state, it is assumed that the agent is able to produce a chemical, either component (-1) or $(+1)$, in dependence on the actual value of the internal parameter. We note that the agent's ability to produce the chemical in general depends on another internal parameter, namely, the internal energy depot that may set limits to the agent's activities. In this model, however, it is assumed that the internal energy depot is always sufficiently balanced, thus its influence shall be neglected here. The agent's chemical production rate, $s_i(\theta_i,t)$, is assumed as follows:

$$s_i(\theta_i,t)=\frac{\theta_i}{2}[(1+\theta_i)s_{+1}^0\exp\{-\beta_{+1}(t-t_{n+}^i)\} -(1-\theta_i)s_{-1}^0\exp\{-\beta_{-1}(t-t_{n-}^i)\}]. \quad (3)$$

Equation (3) means that the agent is not active, as long as $\theta_i=0$, which means before it hits one of the nodes the first time. After that event, the agent begins to produce either component $(+1)$ if $\theta_i=+1$, or component (-1) if $\theta_i=-1$. This activity, however, goes down with time, expressed in an exponential decrease of the production rate. Here, s_{+1}^0, s_{-1}^0 are the initial production rates and β_{+1}, β_{-1} are the decay parameters for the production of the chemical components $(+1)$ or (-1). Respectively, t_{n+}^i, t_{n-}^i are the times, when the agent i hits either a node with a positive or a negative potential.

The spatiotemporal concentration of the chemicals shall be described by a *chemical field* $h_\theta(\boldsymbol{r},t)$ consisting either of component $(+1)$ or (-1), which obeys the following equation:

$$\frac{\partial h_\theta(\boldsymbol{r},t)}{\partial t}=-k_\theta h_\theta(\boldsymbol{r},t)+\sum_{i=1}^{N} s_i(\theta_i,t)\,\delta_{\theta;\theta_i}\delta(\boldsymbol{r}-\boldsymbol{r}_i(t)). \quad (4)$$

The first term describes the exponential decay of the existing concentration due to spontaneous decomposition of the chemical, where k_θ is the decomposition rate. The second term denotes the production of the field by the agents. Here, $\delta_{\theta;\theta_i}$ means the Kronecker delta used for discrete variables, indicating that the agents only contribute to the field component that matches their internal parameter θ_i. The δ function $\delta(\boldsymbol{r}-\boldsymbol{r}_i(t))$ means that the agents contribute to the field only *locally*, at their current position, $\boldsymbol{r}_i$. Diffusion of the chemical substances is not considered here.

The *effective* field, $h^e(\boldsymbol{r},t)$, is a specific function of the different components of the field, Eq. (4). It should influence the movement of the agents according to the overdamped Langevin Eq. (1) and dependent on their current internal parameter, θ_i, as follows:

$$\frac{\partial h^e(\boldsymbol{r},t)}{\partial \boldsymbol{r}}=\frac{\theta_i}{2}\left[(1+\theta_i)\frac{\partial h_{-1}(\boldsymbol{r},t)}{\partial \boldsymbol{r}}-(1-\theta_i)\frac{\partial h_{+1}(\boldsymbol{r},t)}{\partial \boldsymbol{r}}\right]. \quad (5)$$

Equation (6) summarizes the nonlinear feedback between the field and the agents, as given by Eqs. (3), (4), and (5),

θ_i	$\nabla_i h^e(\boldsymbol{r},t)$	$s_i(\theta_i,t)$
0	0	0
$+1$	$\nabla_i h_{-1}(\boldsymbol{r},t)$	$s_i(+1,t)$
-1	$\nabla_i h_{+1}(\boldsymbol{r},t)$	$s_i(-1,t)$.

(6)

Before presenting computer simulations, we would like to summarize our model of network formation that is introduced here in terms of an *agent-based* approach. Each agent is active in the sense that it can (i) move, (ii) produce locally one out of two different chemical cues, and (iii) respond to local gradients of these different chemicals. The actions of all agents are coupled indirectly via an effective field that comprises two different chemical components. The agent activities further depend on an internal parameter θ_i that allows to describe a different ''behavior.'' Our model assumes that agents with an internal state $\theta_i=0$ do not contribute to the field and are not affected by the field. They simply move like Brownian particles. Agents with an internal state $\theta_i=+1$ contribute to the field by producing the chemical cue $(+1)$, while they are affected by the part of the field that is determined by chemical (-1). On the other hand, agents with an internal state $\theta_i=-1$ contribute to the field by producing chemical (-1) and are affected by the part of the field, which is determined by component $(+1)$. Moreover, if the agent hits one of the nodes, the internal state can be switched due to Eq. (2). Hence, the agent begins to produce a different chemical while being affected by the opposite potential. Precisely, at one time the agent does *not* respond to the gradient of the same field component, which it contributes to via producing a chemical.

As the result of this nonlinear feedback between the Brownian agents and the effective field generated by them, we can observe the formation of macroscopic structures shown in the following section.

III. SIMULATION RESULTS OF NETWORK FORMATION

For the computer simulations, a triangular lattice with periodic boundary conditions was used. Further, we have assumed that the parameters describing the production and decay of the chemical are the same for both components,

$$s_{+1}^0=s_{-1}^0=s_0, \quad k_{+1}=k_{-1}=k_h, \quad \beta_{+1}=\beta_{-1}=\beta. \quad (7)$$

The agents start initially at random positions and with the internal parameter $\theta_i(t=0)=0$. For the evolution of the network, we evaluate the sum $\hat{h}(\boldsymbol{r},t)$ of the two field components generated by the agents. For the plots, however, we have to match these values with a *gray scale* of 256 values, which is defined as follows:

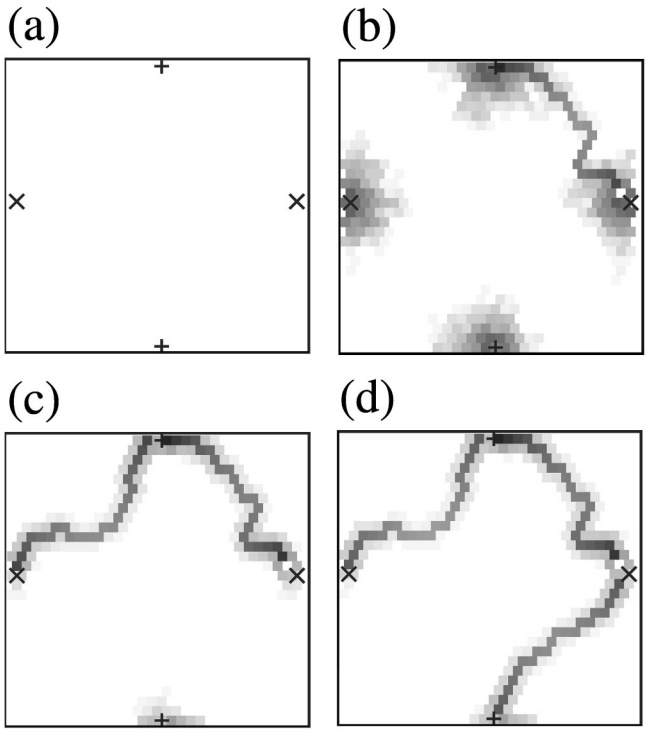

FIG. 2. Formation of links among four nodes. The plots show the spatial concentration $c(\boldsymbol{r},t)$, Eq. (8) on a lattice of size $A=30\times30$ for four different times: (a) initial state, (b) after 100 simulation steps, (c) after 1000 simulation steps, (d) after 4500 simulation steps. Parameters: $N=450$ agents, $s_0=25\,000$, $k_h=0.01$, $\beta=0.2$, $k_h=0.01$.

$$c(\boldsymbol{r},t)=255\left[1-\log_{10}\left(1+9\frac{\hat{h}(\boldsymbol{r},t)-\hat{h}_{min}(t)}{\hat{h}_{max}(t)-\hat{h}_{min}(t)}\right)\right],$$

$$\hat{h}(\boldsymbol{r},t)=h_{+1}(\boldsymbol{r},t)+h_{-1}(\boldsymbol{r},t). \tag{8}$$

This means that the highest actual value, $\hat{h}_{max}(t)$, always refers to *black* ($c=0$), whereas the actual minimum value, $\hat{h}_{min}(t)$ encodes *white* ($c=255$). Both extreme values change in course of time, therefore each snapshot of the time series presented has its own value mapping.

As a first example, we show the evolution of the connections among four nodes (Fig. 2). In the course of time agents that have by chance discovered a node (this way going over into an active state) begin to perform a *directed motion* between the different nodes. Eventually, a link appears which can be clearly distinguished from the surrounding.

Figure 2 would suggest that in the course of time all nodes with an opposite potential should be connected. This, however, is not the case because the existing connections cause a *screening effect* that forces the agents to move along existing connections rather than making up new ones. This screening effect becomes more obvious when the number of nodes is increased. Figure 3 shows the time evolution of a network, which should connect 40 nodes (cf. Fig. 1). We see [29] that in the course of time the agents aggregate along the connections, which results in higher agent concentrations

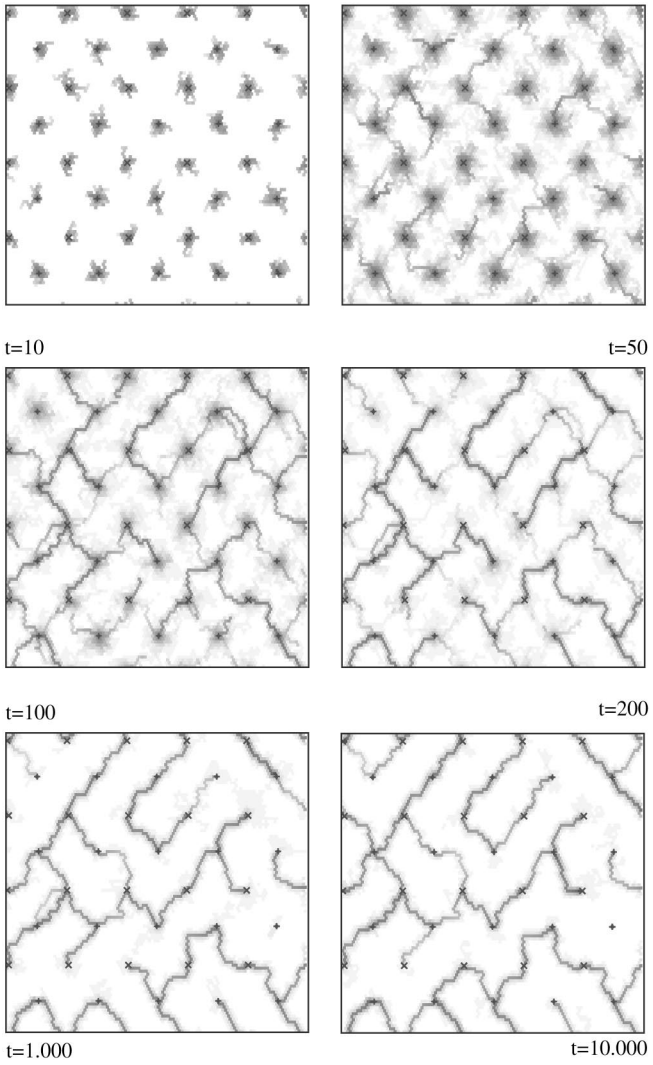

FIG. 3. Time series of the evolution of a network. The plots show the spatial concentration $c(\boldsymbol{r},t)$, Eq. (8) on a lattice of size $A=100\times100$ for different times (in simulation steps). The initial state is shown in Fig. 1. Parameters: $N=5000$ agents, $s_0=10\,000$, $k_h=0.03$, $\beta=0.2$.

and in higher fields along the connections. The self-assembling network is created very fast and remains stable in the long run.

The time series of Fig. 3 indicates that for the formation of the network a *transient stage* exists, during which new nodes are discovered and new connections appear. After the transient time t_{tr}, however, the existing links are only stabilized, with small possible fluctuations. In order to get an estimate of the transient time, we have evaluated the total fraction $x_\theta(t)=N_\theta(t)/N$ of agents that currently have the internal parameter θ. The result shown in Fig. 4 is based on the simulations of Fig. 3. It indicates that after $t\approx1500$ simulation steps every agent has found at least one node by chance, thus changing its internal parameter either to $(+1)$ or to (-1). Further, after this time the share between these internal parameters is almost equally balanced, with slight fluctuations around $x_\theta=0.5$, dependent on the actual position of the agents.

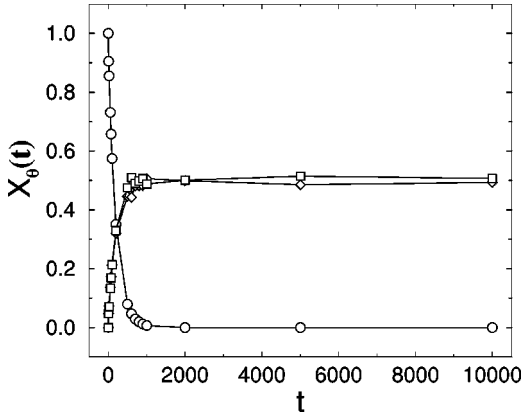

FIG. 4. Fraction x_θ of agents with the internal parameter θ vs time (in simulation steps). ($\bigcirc$), $\theta=0$; ($\diamond$), $\theta=+1$; ($\square$), $\theta=-1$. The values are obtained from the simulation, Fig. 3. Initial conditions: $x_0=1$, $x_{\pm 1}=0$.

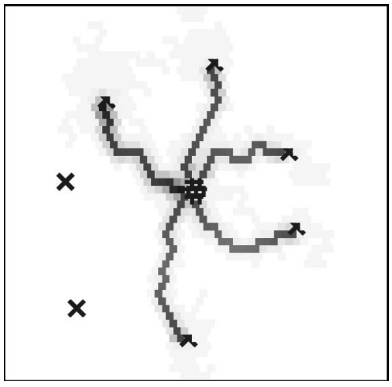

FIG. 5. Formation of links between a center ($z_-=1$) and surrounding nodes ($z_+=7$). The plot shows the spatial concentration $c(\boldsymbol{r},t)$, Eq. (8) on a lattice of size $A=50\times 50$ after $t=10\,000$ simulation steps. Parameters: $N=2000$ agents, $s_0=20\,000$, $k_h=0.02$, $\beta=0.2$.

In Ref. [15], we have shown that the equations for $x_\theta(t)$ for the ensemble average read explicitly

$$x_0(t)=\exp\left\{-D_n\frac{z_++z_-}{A}t\right\}, \tag{9}$$

$$x_\theta(t)=\frac{z_\theta}{z_++z_-}\left(1-\exp\left\{-D_n\frac{z_++z_-}{A}t\right\}\right),$$

$$\theta\in\{-1,+1\}.$$

In the asymptotic limit, the fraction x_θ is determined by the appropriate number of nodes, which change the internal state of the agents into θ. The transient time t_{tr} can be estimated by assuming that the difference between $x_\theta(t)$ and the stationary value x_θ^{stat} should be smaller than a certain value, κ,

$$t_{tr}\geqslant\frac{A}{D_n z}\ln\left(\frac{1}{\kappa}\right). \tag{10}$$

For $D_n=1$, we find for $\kappa=10^{-2}$ a transient time of $t_{tr}=1150$, and for $\kappa=10^{-3}$ $t_{tr}=1700$, which is in good agreement with the results of the computer simulations. After that time, the assembled network should remain almost stable.

Patterns like the network shown are intrinsically determined by the history of their creation. It means that irreversibility and early symmetry breaking play a considerable role in the determination of the final structure. The location of the different nodes acts more or less as a boundary condition for the structure formation which sets limits to the achievable structures, but does not determine the way of connecting the different nodes.

Despite the fact that in Fig. 3 almost all nodes are connected by at least one link, only some out of all possible connections have been realized. In particular, only nearest neighbor nodes with opposite potentials are connected. This is partly due to the screening effect that makes longer connections an unlikely event, but also indicates that, between two nodes, a maximum distance $L^\star$ exists which agents are able to connect. An approximation for this critical distance is given in Ref. [16].

Eventually, we note that the network formation is not restricted to regular or symmetric distributions of nodes. Figure 5 shows a simulation where different nodes are connected with a center.

In Ref. [8] we have also discussed a different variant of the model to demonstrate its flexibility in connecting additional nodes to the network, or disconnecting obsolete ones. Further, in Ref. [16] we have shown that the switching behavior between a connected and a disconneced state can be very short, which would allow the construction of a dynamic switch.

IV. ESTIMATION OF THE NETWORK CONNECTIVITY

A. Definition of connectivity

In order to characterize a network, one of the most important questions is whether two nodes k and l are connected or not. In the model considered, a connection is defined in terms of the chemical field $\hat{h}(\boldsymbol{r},t)$ produced by the agents. During the first stage of the network formation, the agents have randomly visited almost every lattice site before their motion turned into a bound motion between the nodes. Therefore, the field $\hat{h}(\boldsymbol{r},t)$ has a nonzero value for almost every $\boldsymbol{r}$, which exponentially decays, but never vanishes. Hence, in order to *define a connection* in terms of $\hat{h}(\boldsymbol{r},t)$, we have to introduce a *threshold value* h_{thr}, which is the minimum value considered for a link. More precisely, a *connection* between two nodes k and l should only exist if there is a path $a\in A$ between k and l along which the actual value of the field is larger than the threshold value,

$$\hat{h}(a,t)>h_{thr}\quad\text{for}\quad a\in A. \tag{11}$$

Such a definition does not necessarily assume that the connection has to be a *direct link*. Instead, it could be any path a,

which may also include other nodes, as long as the value $\hat{h}(a,t)$ along the path is above the threshold.

We want to define the *local connectivity* E_{lk} as follows:

$$E_{lk}=\begin{cases}1 & \text{if } k \text{ and } l \text{ are connected by a path } a\in A\\ & \text{along which } \hat{h}(a,t)>h_{thr}\\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

We note that the connectivity E_{lk} does not change if two nodes k and l are connected by more than one path.

If we consider a number of z nodes, then the *global connectivity* E that refers to the whole network is defined as follows:

$$E=\frac{\sum_{k=1}^{z}\sum_{l>k}^{z}E_{lk}}{\sum_{k=1}^{z}\sum_{l>k}^{z}1}=\frac{2}{z(z-1)}\sum_{k=1}^{z}\sum_{l>k}^{z}E_{lk}. \tag{13}$$

Depending on the configuration of nodes, there may be numerous different realizations for the connections, which result in the same connectivity E.

B. Estimation of the threshold value

In order to use the definition for the connectivity to evaluate the simulated networks, we first have to define the threshold value h_{thr}. This should be the *minimum value* of $\hat{h}(\boldsymbol{r},t)$ along a *stable connection* between two nodes. For our estimations, we treat the connection between two nearest neighbor nodes k and l as a *one-dimensional* structure, where x is now the space coordinate, and L the linear distance between the two nodes k and l. The node at $x=0$ should have a positive potential $V=+1$, while the node at $x=L$ has a negative potential $V=-1$,

$$0\leqslant x\leqslant L,\quad V(0)=+1,\quad V(L)=-1. \tag{14}$$

We assume that a stable connection exists if both field components $h_\theta(x,t)$ have reached their stationary values,

$$\frac{\partial h_\theta(x,t)}{\partial t}=-k_\theta h_\theta(x,t)+\sum_i s_i(\theta,t)\,\delta(x-x_i)=0. \tag{15}$$

Of course, we do not know how many agents are actually on the connection between k and l. For our estimations we have to bear in mind that h_{thr} should determine the *lower limit* of the possible values of $\hat{h}$, therefore it is justified to assume the worst case, which means that the local number of agents at a specific location is just given by the *average agent density* $\overline{n}=N/A$, where N is the total agent number and A is the surface size. Further, we found in the computer simulations (cf. Fig. 4), that in the long-time limit the agents are equally distributed between the two internal states, $\theta\in\{+1,-1\}$. Hence, we assume that on any location along the connection there are $n_\theta=\overline{n}/2$ agents in state θ. Using this lower limit for the agent number, the δ function in Eq. (15) can be replaced by $\overline{n}/2=N/2A$ in the continuous limit.

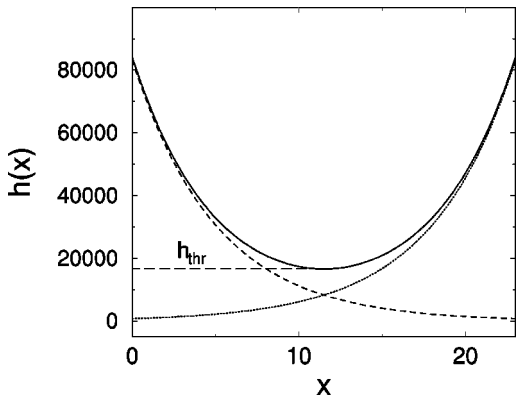

FIG. 6. Stationary solutions for $h_{+1}(x)$, Eq. (19) $(\cdots)$, $h_{-1}(x)$ $(--)$, and $\hat{h}(x)$, Eq. (20) (——) vs spatial coordinate x. $h(x)$ is measured in units of $[s_0]$ per length and time unit. $L=L_{max}/2$, Eq. (24). The threshold h_{thr} is defined as the minimum value of $\hat{h}(\boldsymbol{r},t)$ in the stationary limit. For the parameters see Fig. 3.

Further, we consider that the agents move along the x coordinate with a constant velocity (which also matches with the assumption $\dot{v}\approx 0$ of the overdamped Langevin equation),

$$v=|\boldsymbol{v}|=\frac{|x|}{t},\quad 0\leqslant x\leqslant L. \tag{16}$$

This allows us to replace t in the time dependent production rate, $s_i(\theta,t)$. With these simplified assumptions and the conventions of Eq. (7), Eq. (15) reads for component $\theta=+1$,

$$\frac{\partial h_{+1}(x,t)}{\partial t}=-k_h h_{+1}(x,t)+\frac{\overline{n}}{2}s_0\exp\left\{-\beta\frac{x}{v}\right\}. \tag{17}$$

Integration of Eq. (17) yields with $h_{+1}(x,t=0)=0$,

$$h_{+1}(x,t)=\frac{\overline{n}}{2}\frac{s_0}{k_h}\exp\left\{-\beta\frac{x}{v}\right\}(1-\exp\{-k_h t\}). \tag{18}$$

Eventually, for $t\to\infty$ we find from Eq. (18) the stationary solution

$$h_{+1}(x)=\frac{\overline{n}}{2}\frac{s_0}{k_h}\exp\left\{-\frac{\beta}{v}x\right\}. \tag{19}$$

The remaining field component $h_{-1}(x',t)$ should have the same stationary solution as Eq. (19), with $x'=L-x$. The resulting total field $\hat{h}(x,t)$ reads in the stationary limit (cf. Fig. 6)

$$\hat{h}(x)=h_{+1}(x)+h_{-1}(L-x)$$
$$=\frac{\overline{n}}{2}\frac{s_0}{k_h}\left[\exp\left\{-\frac{\beta}{v}x\right\}+\exp\left\{-\frac{\beta}{v}(L-x)\right\}\right], \tag{20}$$
$$0\leqslant x\leqslant L.$$

The threshold h_{thr} should be defined as the minimum of $\hat{h}(x)$, which yields for $L/2$. As the result, we find

$$h_{thr}=\hat{h}\left(\frac{L}{2}\right)=\frac{N}{A}\frac{s_0}{k_h}\exp\left\{-\frac{\beta}{v}\frac{L}{2}\right\}. \tag{21}$$

Here, the threshold value is a function of the mean agent density $\bar{n}$, the parameters s_0, k_h, and β, and the distance between the two nodes, L. However, due to the decay of the field (k_h) and the decreasing production rate with time (β), the agents are only able to link nodes that are at a distance closer than a critical distance $L^\star$, i.e., Eq. (21) makes sense only for $L<L^\star$.

In order to get an estimation for $L^\star$, we assume that a minimum production rate s_{min} exists, which is in the given model the smallest possible amount of chemical released by the agent (naturally, it could be a molecule, if s is measured in molecule numbers). With t_0 being the time when the agent hits the node, we get from

$$s_{min}=s_0\exp\{-\beta(t-t_0)\} \tag{22}$$

the maximum time t_{max} after which the production is *negligible*,

$$t_{max}=\frac{1}{\beta}\ln\left\{\frac{s_0}{s_{min}}\right\}. \tag{23}$$

We can now discuss the case that the agent moves straight with a constant velocity, without changing its direction. Then the maximum distance crossed before the contribution to the field is negligible would be

$$L_{max}=v\,t_{max}=\frac{v}{\beta}\ln\left\{\frac{s_0}{s_{min}}\right\}. \tag{24}$$

On the contrary, if we assume that the agent moves like a random walker, the average distance reached after t simulation steps, is given by the mean displacement, $\Delta R=\sqrt{2d\,D_n t}$, which yields for $d=2$,

$$L_{av}=\sqrt{2\,D_n t_{max}}=\sqrt{\frac{2\,D_n}{\beta}\ln\left\{\frac{s_0}{s_{min}}\right\}}. \tag{25}$$

The real maximum distance that can be connected by one agent in the considered model is of course between these limits. We have found [16] that $L_{max}/2$ is a reasonable estimate for $L^\star$,

$$\sqrt{\frac{2D_n}{v}L_{max}}<L^\star\approx\frac{L_{max}}{2}<L_{max}. \tag{26}$$

Using this approximation, we find with Eq. (21) and Eq. (24) eventually the estimate for the threshold,

$$h_{thr}=\frac{N}{A}\frac{s_0}{k_h}\left(\frac{s_{min}}{s_0}\right)^{1/4}. \tag{27}$$

Provided the set of parameters used for the simulations, we find for the threshold the value $h_{thr}=1.7\times10^4$, which is approximately $h_{thr}\approx2s_0$. We note again that this is an estimate that might give a rather high value because of the assumed worst case conditions. On the other hand, it ensures that values for $\hat{h}(a,t)$ above the threshold *really* represent a *stable* connection a.

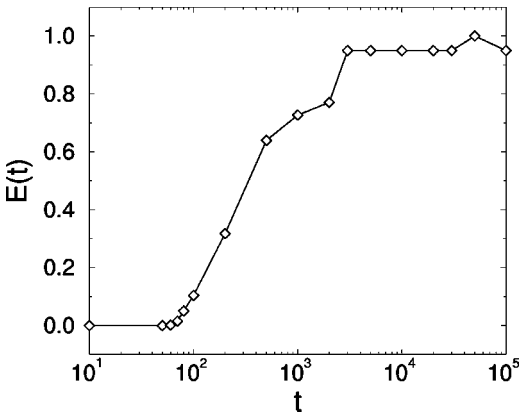

FIG. 7. Network connectivity E Eq. (13) vs time t (in simulation steps), calculated from the series of Fig. 3.

C. Results of computer simulations

After these theoretical considerations, we are now able to calculate the connectivity E, Eq. (13), for the network simulated in Fig. 3. Figure 7 shows the increase of the connectivity in the course of time. In agreement with the visible evolution of the network presented in Fig. 3, the following three different stages can be clearly distinguished.

(1) An *initial* period ($t<10^2$), where no connections yet exist.

(2) A *transient* period ($10^2<t<10^4$), where the network establishes.

(3) A *saturation* period ($t>10^4$), where almost all nodes are connected, and only small fluctuations in the connectivity occur.

Figure 7 results from the single realization of the network shown in Fig. 3, in the average we find certain fluctuations in the connectivity due to stochastic influences that affect the formation of the network during the transient period. This leads us to the question: on what parameters the connectivity of the network depends? There are of course the parameters affecting the production and decay of the two different chemical cues, s_0/k_h, β, and thus the positional information available to the agents. Another important parameter is the average agent density $\bar{n}=N/A$. If it is too low, the links will not be established properly, either because not all nodes have been detected during the transient period or because there are not enough agents to maintain the links sufficiently. Figure 8 shows the average connectivity $\langle E\rangle$ dependent on the density of agents, $\bar{n}$.

Here, we clearly see that below a critical density the connectivity is almost zero because not enough agents are available to establish the connections. On the other hand, above a certain density the connectivity reaches a saturation value that could be also below 1, as Fig. 8 shows. Hence, at this point an increase of the number of agents does not necessar-

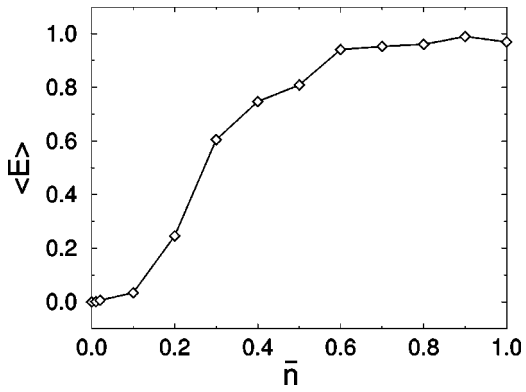

FIG. 8. Network connectivity $\langle E \rangle$ Eq. (13), averaged over five simulations vs mean density of agents, $\bar{n}$ (number of agents per lattice size, N/A). For further parameters see Fig. 3.

ily result in the establishment of more links. This is caused by the *screening effect* already mentioned in Sec. III, which eventually concentrates all agents to move along the established links.

A third important impact on the establishment of the network results from parameter relation between α_i and ε_i which influence the agent's motion according to Eq. (1). If both the response to the field, α_i, and the sensitivity $\omega_i \propto 1/\varepsilon_i$ are low, the agent nearly behaves as a random particle. On the other hand, a strong response or a high sensitivity may result in a decrease of stochastic influences, and the agent pays more attention to the effective field that guides its motion. This in turn increases the screening effect and may prevent the agent from discovering unknown nodes, thus we can expect an optimal range of these parameters for an efficient network formation. This has been investigated in more detail in a subsequent paper [15].

V. CONCLUSIONS

In this paper, we have proposed an agent-based model that shows the *self-assembling* of networklike structures between arbitrary nodes. Different from network models that start with the assumtion of a known set of nodes to be linked in a straightforward manner, we have addressed in our model the question of how to connect a set of nodes *without* using prior information about their spatial locations or preexisting long-range attraction forces. This would need to solve two problems simultaneously: (i) the detection of the appropriate nodes, and (ii) the establishment of one (or many) stable links between them.

As we have noted in the Introduction, this is a scientific problem of relevance in different areas, including the emergence of neural connections. In the latter case, positional information in terms of chemical gradients plays an important role in guiding the neural axons to their destinations. In our model, we have assumed that the positional information about the existence of the nodes does not preexist, but is generated ''on the fly'' by means of Brownian agents while they are moving on the surface. Due to the nonlinear feedback between the detected positional information and the creation of new links, we find the emergence of links between different nodes, along which the Brownian agents perform a directed motion—in this way reinforcing and ''maintaining'' the links.

Different from a *circuitry* for instance, for which the links between the nodes are determined in a *top-down* approach of hierarchical planning, the connections here are created by the agents *bottom up*, in a process of self-organization. As the computer simulations have shown, the model turns out to be very flexible regarding the geometry of the nodes to be connected. Further, the networks created this way are rather *robust* against disturbances. If, for example, a particular link breaks down, the agents would be able to *repair* it by reestablishing the field, or by creating a new one.

The basic feedback mechanism in our model is given by the agent's generation of (two) different kinds of chemical information and the agent's response to gradients of these chemicals dependent on their internal state. This is also known as *chemotaxis*, i.e., the response to (gradients of) chemical substances and is widely found in different levels of biological organization. However, different from other self-wiring models [4,5] we have not assumed that these substances due to their diffusion may have a long-range effect on the moving agents, or may generate attractive *and* repulsive forces. In our model, the chemical information acts only locally. It is stored in an effective field that is sometimes also denoted as a *self-consistent field*, because it is generated by the agents and at the same time also influences their further behavior. From a more general perspective, the effective field plays the role of a communication medium among the agents, i.e., it stores information (external to the agents) for a certain time (determined by the decay rate k_h) and allows access to it under certain conditions—for instance, in the current model an agent can only access information that is at its current position and is labeled differently from the value of the agent's internal parameter. But the agents do not just respond to the information provided in a purely reactive manner, they also actively change it dependent on their internal parameter.

The Brownian agent concept has proven its utility in a number of applications where (positive and negative) local feedback processes play a considerable role, but an internal *evolution* of the agents can be neglected [12]. The concept does not deny its inspiration from statistical physics, using, e.g., generalized Langevin equations for the agents or reaction-diffusion equations for the effective field. Moreover, it purposefully stretches these analogies, in order to apply methods from statistical physics to derive pieces for a formal approach to *multiagent systems* (MAS). This should also include the derivation of a macroscopic dynamics of the MAS based on the agents' (*microscopic*) dynamics, to allow some predictions of the collective behavior and the derivation of critical parameters, etc.

On the first glimpse, agent-based approaches seem to be outside the realms of physics. Therefore, at the end we would like to comment on this. First of all, ''agent'' or particle-based models are also useful in physics if continuous approximations are not appropriate, for example, in cases where only small particle numbers govern the process (e.g., in dielectrical breakdown or filamentary pattern formation).

Here deterministic approaches or mean-field equations are not sufficient to describe the behavior of the system, because the influence of history, i.e., stochastic fluctuations, early symmetry breaks, path dependence, etc., play a considerable role—which can be appropriately captured in particle-based models. Moreover, these models also provide a very efficient way to simulate structure formation processes by solving a large number of coupled ''particle'' equations (such as Langevin equations) instead of integrating a complicated set of coupled partial differential equations [17,18].

In many interdisciplinary applications physics nowadays deals with, for example, in biological physics or econophysics, the basic system entities do not just respond to interaction forces, but also perform certain types of activities, such as active motion or changes of the ''environment'' and further have internal degrees of freedom that allow them to act differently. Therefore, the ''physical'' particle-based approach has been extended towards an *agent-based approach*, where the agents already have an intermediate complexity to allow for certain nontrivial actions or responses. In the model discussed in this paper, the generation of two different chemical cues and the complex response to them are just two examples for such an extension. In addition to the Brownian agent model, other agent-based approaches have been developed based on physical principles, which focus on particular aspects of the complex agent behavior. We just mention here *active walker models* [6,19–22], where a potential can be locally changed by the walker, or *active Brownian particles* [14,18] that deal also with the energetic aspects of active motion. With respect to specific biological phenomena, there is the *communicating walker model* used in the study of complex patterning of bacterial colonies [23], or the *bions model* used in the study of amebae aggregation [24], or the many models of self-driven particles to describe swarming behavior [25–28]. They jointly demonstrate that agent-based models can indeed profit from the methodology and the tools derived in statistical physics.

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[1] L. Wang, Sci. News (Washington, D. C.) **159**, 286 (2001).
[2] Y. Cui and C.M. Lieber, Science **291**, 851 (2001).
[3] D. Mange and A. Stauffer, in *Artificial Life and Virtual Reality*, edited by N. M. Thalmann and D. Thalmann (Wiley, Chichester, England, 1994), pp. 61–72.
[4] R. Segev and E. Ben-Jacob, Phys. Lett. A **237**, 307 (1998).
[5] R. Segev and E. Ben-Jacob, Neural Networks **13**, 185 (2000).
[6] D. Helbing, F. Schweitzer, J. Keltsch, and P. Molnár, Phys. Rev. E **56**, 2527 (1997).
[7] T.O. Crist and J.W. Haefner, J. Theor. Biol. **166**, 315 (1994).
[8] F. Schweitzer, K. Lao, and F. Family, BioSystems **41**, 153 (1997).
[9] M. Tessier-Lavigne and C.S. Goodman, Science **274**, 1123 (1996).
[10] A. Gierer, Proc. R. Soc. London, Ser. B **218**, 77 (1983).
[11] R.W. Sperry, Proc. Natl. Acad. Sci. U.S.A. **50**, 703 (1963).
[12] F. Schweitzer, *Brownian Agents and Active Particles*, Springer Series in Synergetics (Springer, Berlin, 2002).
[13] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. Lett. **80**, 5044 (1998).
[14] F. Schweitzer, in *Stochastic Dynamics*, edited by L. Schimansky-Geier and T. Pöschel (Springer, Berlin, 1997), Vol. 484, pp. 358–371.
[15] F. Schweitzer and B. Tilch (unpublished).
[16] F. Schweitzer and B. Tilch (unpublished).
[17] L. Schimansky-Geier, F. Schweitzer, and M. Mieth, in *Self-Organization of Complex Structures: From Individual to Collective Dynamics*, edited by F. Schweitzer (Gordon and Breach, London, 1997), pp. 101–118.
[18] L. Schimansky-Geier, M. Mieth, H. Rosé, and H. Malchow, Phys. Lett. A **207**, 140 (1995).
[19] F. Schweitzer and L. Schimansky-Geier, Physica A **206**, 359 (1994).
[20] L. Lam, Chaos, Solitons Fractals **6**, 267 (1995).
[21] C.-R. Sheu, C.-Y. Cheng, and R.-P. Pan, Phys. Rev. E **59**, 1540 (1999).
[22] D.R. Kayser, L.K. Aberle, R.D. Pochy, and L. Lam, Physica A **191**, 17 (1992).
[23] E. Ben-Jacob, O. Schochet, A. Tenenbaum, I. Cohen, A. Czirók, and T. Vicsek, Nature (London) **368**, 46 (1994).
[24] D. Kessler and H. Levine, Phys. Rev. E **48**, 4801 (1993).
[25] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. E **64**, 021110 (2001).
[26] A. Czirók and T. Vicsek, Physica A **281**, 17 (2000).
[27] A. Mikhailov and D.H. Zanette, Phys. Rev. E **60**, 4571 (1999).
[28] H. Levine, W.-J. Rappel, and I. Cohen, Phys. Rev. E **63**, R017101 (2000).
[29] A video of these computer simulations can be found at http://www.ais.fhg.de/~frank/network.html